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arXiv:hep-th/0703010v1 1 Mar 2007

Paths for $\mathcal{Z}_k$ parafermionic models

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February 2007

Abstract

We present a simple bijection between restricted (Bressoud) lattice paths and RSOS paths in regime II. Both types of paths describe states in $\mathcal{Z}_k$ parafermionic irreducible modules. The bijection implies a direct correspondence between a RSOS path and a parafermionic state in a quasi-particle basis.

1 Introduction

1.1 The two path descriptions of the parafermionic $\mathcal{Z}_k$ models

The analysis of the RSOS model by the corner-transfer-matrix method has led to the expression of the local-height probability in terms of a certain one-dimensional configuration sum [2]. Every configuration appearing in the sum is naturally associated to a (RSOS) path. In the infinite-length limit, this path is in correspondence with a state of an irreducible module (specified by the boundary conditions on the path) of some conformal field theory, and the sum over all paths (with fixed boundary conditions) reproduces the character of this module [7, 8]. In regime III, these are the characters of the minimal models $\mathcal{M}(k+1,k+2)$ [3, 9], while those of the parafermionic $\mathcal{Z}_k$ models [21] are recovered in regime II. The two regimes are distinguished by the way the paths are weighted. The parameter k specifying these models is related to the original parameter r of [2] by $r=k+2$ [13].

The path representation leads naturally to a fermionic form of the characters. The fermionic evaluation of the RSOS configuration sums has been worked out in [19, 20] in their finitized version. From this, the fermionic form of the irreducible characters of the minimal models and (by a duality transformation to be reviewed below) those of the parafermionic models are recovered by taking the infinite-path limit.

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Somewhat remarkably, there is another path description for the parafermionic states: these can be represented by the $(k-1)$-restricted lattice paths described in [5]. The most direct way of seeing this relationship is to notice that these paths are actually in one-to-one correspondence with partitions $(\nu_1, \cdots, \nu_m)$ (where m is the total length of the path, to be defined below) satisfying the following condition [6, 5, 17]:

$$\nu_i - \nu_{i+k-1} \geq 2 \,. \tag{1}$$

But this condition is precisely the combinatorial constraint governing the quasi-particle basis of states of the parafermionic $\mathcal{Z}_k$ models [18, 15]. The conformal dimension of a parafermionic state is the weight of the corresponding path (also defined below), up to a prescribed correction term accounting for the fractional dimension of the parafermionic modes.

The parafermionic states can thus be described by two superficially quite different types of paths. Here we show that these paths are essentially the same. In other words, we provide a very simple bijection between the RSOS and (Bressoud) lattice paths. Since the latter are in one-to-one correspondence with states in a quasi-particle description of the parafermionic modules, we end up with a similar correspondence between a RSOS path in regime II and a parafermionic state in a quasi-particle basis.

1.2 A duality transformation for RSOS paths

We recall in this section the duality transformation that relates the $\mathcal{M}\,(k+1,k+2)$ and the $\mathcal{Z}_k$ models within the framework of their RSOS description [11]. This duality allows us to translate results derived in the context of the unitary minimal models [19, 20] to the parafermionic case.

The configurations referred to in the terminology 'configuration-sum' are those of height variables σ_i that take values in the set $\{0,1,\cdots,k\}$, and i ranges from 0 to L. Adjacent heights are subject to the restriction $|\sigma_i - \sigma_{i+1}| = 1$. The two boundary values are fixed: let us set for definiteness $\sigma_0 = \sigma_L = 0$. The configuration-sum then takes the form

$$X(q) = \sum_{\substack{\sigma_1,\cdots,\sigma_{L-1}=0 \\ |\sigma_i-\sigma_{i+1}|=1 \\ \sigma_0=\sigma_L=0}}^{k} q^{\sum_{i=1}^{L-1} \tilde{w}(i)} \,. \tag{2}$$

If we plot all the points (i, σ_i) of a given configuration and link adjacent points by a straight line we obtain a path, called a RSOS path. The weight function $\tilde{w}(i)$ depends upon the regime under consideration. We have:

$$\begin{aligned} &\text{if } i \text{ is not an extremum of the path :} \qquad && \tilde{w}_{\rm III}(i) = \frac{i}{2} \quad \text{and} \quad \tilde{w}_{\rm II}(i) = 0 \,, \\ &\text{if } i \text{ is an extremum of the path :} \qquad && \tilde{w}_{\rm III}(i) = 0 \quad \text{and} \quad \tilde{w}_{\rm II}(i) = \frac{i}{2} \,. \end{aligned} \tag{3}$$

The two weight functions are thus related by

$$\tilde{w}_{\rm III}(i) = \frac{i}{2} - \tilde{w}_{\rm II}(i) \,. \tag{4}$$

This implies that

$$X_{\rm III}(q) = q^{L(L-1)/4}\, X_{\rm II}(q^{-1}) \,. \tag{5}$$

This is the announced duality: up to an irrelevant factor, the two configuration sums are related by $q \to q^{-1}$ [11]. An operation that increases the weight in one case, decreases it in the dual case. In particular, the minimal-weight configuration corresponding to the finitized $\mathcal{M}\,(k+1,k+2)$ model is mapped to the maximal-weight configuration for the finitized $\mathcal{Z}_k$ model.

1.3 Outline

The bijection between lattice paths and RSOS paths is worked out in various steps in Sec. 2. To keep the presentation as simple as possible, this correspondence is first presented for paths describing the vacuum parafermionic module. In Sec. 3.1, we point out that the bijection between a RSOS path and a parafermionic state entails a new expression for the weight of a RSOS path. This novel weight formula is proven directly. The rest of this section is concerned with a simple construction of the generating function of the RSOS path for a fixed length, that is, a derivation of the finitized parafermionic vacuum character. The formulae describing a generic module are presented in the final section.

2 Bijection between lattice paths and RSOS paths

2.1 The multiple-partition basis

In [16] (see also [12]), a new quasi-particle-type basis for the states describing the irreducible modules of the $\mathcal{Z}_k$ models has been presented. It is formulated in terms of the modes of the $k-1$ different parafermionic fields $\psi^{(j)}$, with $1 \leq j \leq k-1$, of conformal dimension [21]:

$$h_{\psi^{(j)}} = \frac{j(k-j)}{k}. \tag{6}$$

The modes of $\psi^{(j)}$ are denoted by $A^{(j)}_{-n+j(j+q)/k}$, where n is an integer. In this notation, the conformal dimension of $A^{(j)}_u$ is $-u$. The fractional part of the mode, $j(j+q)/k$ involves the parafermionic charge q of the state on which the mode acts [21]. The charge of $A^{(j)}_u$ is normalized to $2j$ and that of the vacuum state is 0.

Consider for simplicity the states in the vacuum module. These are described by the set of all strings of modes ordered as follows:

$$A^{(1)}_{-\lambda^{(1)}_1+\frac{(1+\hat{q})}{k}} \cdots A^{(1)}_{-\lambda^{(1)}_{m_1}+\frac{(1+\hat{q})}{k}} \cdots A^{(k-1)}_{-\lambda^{(k-1)}_1+\frac{(k-1)(k-1+\hat{q})}{k}} \cdots A^{(k-1)}_{-\lambda^{(k-1)}_{m_{k-1}}+\frac{(k-1)(k-1+\hat{q})}{k}} |0\rangle \,, \tag{7}$$

where $\hat{q}$ is an operator which gives the charge of the string at its right. For instance,

$$\hat{q}(A^{(i)}_u A^{(j)}_v|0\rangle) = 2(i+j)+0. \tag{8}$$

It is not difficult to sum the contribution of the fractional part of the modes to the total conformal dimension of such a string. It actually depends only upon its total charge and not its particular composition [14]. It can thus be evaluated quite simply by replacing all higher-charge modes $A^{(j)}$ with $j>1$ by j copies of $A^{(1)}$. The resulting string has then a total of m $A^{(1)}$ operators, with

$$m = \sum_{j=1}^{k-1} j\, m_j \,, \tag{9}$$

where m_j is the total number of modes $A^{(j)}$ in the original string. The total 'fractional dimension' is then

$$h_{\text{frac}} = \sum_{i=0}^{m-1} \frac{(1+2i)}{k} = \frac{m^2}{k}. \tag{10}$$

The dimension of the state (7) is

$$h = \sum_{j=1}^{k-1} \sum_{l=1}^{m_j} \lambda_l^{(j)} - h_{\text{frac}}. \tag{11}$$

Let us return to (7): we still need to specify the constraints on the 'integral part' of the modes, the $\lambda_l^{(j)}$, for these states to form a basis. These conditions are [16]:

$$\lambda_l^{(j)} \geq \lambda_{l+1}^{(j)} + 2j\,, \tag{12}$$

as well as

$$\lambda_{m_j}^{(j)} \geq j + 2j(m_{j+1} + \cdots + m_{k-1})\,. \tag{13}$$

Since the mode label $\lambda_l^{(j)}$ keeps track of the charge of the corresponding parafermionic mode via its upper-index and the fractional part of the corresponding mode is uniquely recovered from the position of the parafermionic mode in the string (which is itself fixed by the two labels l and j), one can rewrite the sequence (7) more compactly as

$$\lambda_1^{(1)} \cdots \lambda_{m_1}^{(1)} \lambda_2^{(2)} \cdots \lambda_{m_2}^{(2)} \cdots \lambda_1^{(k-1)} \cdots \lambda_{m_{k-1}}^{(k-1)}\,. \tag{14}$$

This sequence is equivalent to the set of $k-1$ ordered partitions of respective lengths $m_1, \cdots, m_{k-1}$, that is,

$$\Lambda^{[k-1]} = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(k-1)}) \qquad \text{with} \qquad \lambda^{(j)} = (\lambda_1^{(j)}, \cdots, \lambda_{m_j}^{(j)})\,. \tag{15}$$

2.2 $(k-1)$-restricted lattice paths

A $(k-1)$-restricted integer lattice path is defined as a sequence of integral points (x, y) within the strip $x \geq 0$ and $0 \leq y \leq k-1$, with adjacent points linked by NE, SE or WE (i.e., horizontal) segments, the latter being allowed only if it lies on the x-axis [5]. There is no definite notion of length for such a path but the final point is forced to be on the x-axis. For lattice paths pertaining to the vacuum module, we also require the initial point to be on the x-axis.

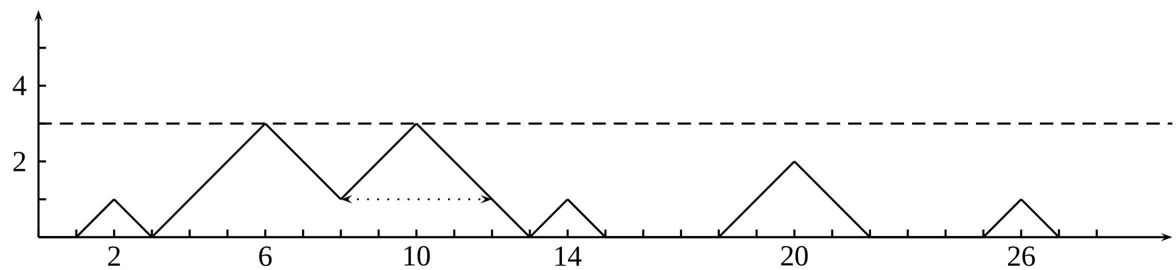
The weight w of a lattice path is the sum of the x-coordinate of all its peaks:

$$w = \sum_{x=1}^{L-1} w(x) \qquad \text{where} \qquad w(x) = \begin{cases} x & \text{if } x \text{ is the position of a peak,} \\ 0 & \text{otherwise}\,. \end{cases} \tag{16}$$

The charge (called the relative height in [5]) of a peak with coordinates (x, y) is the largest integer c such that we can find two points $(x', y-c)$ and $(x'', y-c)$ on the path with $x' < x < x''$ and such that between these two points there are no peaks of height larger than y and every peak of height equal to y has weight larger than x [4]. The charge of a peak is bounded by $1 \leq j \leq k-1$. For instance, in Fig. 1, where $k = 4$, the charge of the peaks, read from left to right, are 1, 3, 2, 1, 2 and 1. A dotted line indicates the line from which the height must be measured to give the charge. Another example is provided by Fig. 2 (with now $k = 5$). There again, dotted lines provide a direct evaluation of the charges. The total charge of a path is the sum of the charges of all its peaks.

Such a path is completely specified by the sequence of its peaks, together with their respective charges. To prepare the ground for the relation between a path and a parafermionic state, we will understand that the sequence of peaks in a path is to be read from right to left. The peak data (x_n, c_n), where x_n is the x-coordinate and c_n the charge, will be written more compactly in the form $x_n^{(c_n)}$. We will also refer to $x_n^{(c_n)}$ as a cluster of charge c_n and weight x_n. For example, the path of in Fig. 1 corresponds to the sequence $26^{(1)}\, 20^{(2)}\, 14^{(1)}\, 10^{(2)}\, 6^{(3)}\, 2^{(1)}$.

Figure 1: An example of lattice path for $k = 4$.



The basic characteristics of a path are captured by the following conditions. The first is that a peak of charge j has minimal x-coordinate j. The second, referred to as the 'path condition', is the following [17]. If between two peaks $x^{(i)}$ and $x'^{(j)}$ there are peaks all with charge lower than $\min(i, j)$ and whose total charge sums to c, then

$$x - x' \geq r_{ij} + \delta_{i>j} + 2c \,, \tag{17}$$

where r_{ij} stands for

$$r_{ij} = 2\min\,(i, j) \,, \tag{18}$$

and $\delta_b = 1$ if b is true and 0 otherwise. The case $c = 0$ describes the minimal separation between two adjacent peaks: if $i > j$, it is $2j+1$, while if $i \leq j$, it is $2i$. Summing up, a $(k-1)$- restricted lattice path, denoted as $\mathrm{LP}^{[k-1]}$ is an ordered sequence of clusters, say s,

$$\mathrm{LP}^{[k-1]} = x_1^{(c_1)} \cdots x_s^{(c_s)} \,, \tag{19}$$

satisfying:

$$1 \leq c_n \leq k-1, \qquad x_s \geq c_s, \qquad x_n > x_{n+1}, \qquad \text{and (17) holds.} \tag{20}$$

In the cluster terminology, a multiple partition like (15) also has the form of an ordered sequence of clusters, but ordered in increasing value of the charge. This suggests that a multiple partition is nothing but the rearrangement of the sequence of clusters defining a path. This calls for a reordering rule for clusters. For this, we introduce a formal exchange operation that describes the interchange of two adjacent clusters $x^{(i)}x'^{(j)}$ [17]:

$$x^{(i)}x'^{(j)} \quad \stackrel{\text{ex}}{\longrightarrow} \quad (x' + r_{ij})^{(j)}(x - r_{ij})^{(i)} \,, \tag{21}$$

where r_{ij} is defined in (18). This operation preserves the individual values of the charge and also the sum of the weights. Note that after a number of interchanges on a sequence of clusters $\{x_n^{(c_n)}\}$ describing a path, such that $\{x_n^{(c_n)}\} \stackrel{\text{ex}}{\longrightarrow} \{\tilde{x}_n^{(c_n)}\}$, the values $\tilde{x}_n$ are no longer necessarily decreasing and they no longer correspond to peak positions in a modified path. Indeed, (21) typically spoils the path condition (17).

The correspondence between a $(k-1)$-restricted lattice path $\mathrm{LP}^{[k-1]}$ and a multiple partition $\Lambda^{[k-1]}$ is now easily described [17]. Take a path read from right to left and use the exchange relation (21) to reorder the clusters in increasing value of the charge from left to right. The weights of the resulting clusters of charge j then form the parts of the partition $\lambda^{(j)}$. The multiple partition is thus a canonical rewriting of the original path, with

$$s = \sum_{j=1}^{k-1} m_j \,. \tag{22}$$

This gives the link $\mathrm{LP}^{[k-1]} \to \Lambda^{[k-1]}$. For instance, we have

$$26^{(1)}\,20^{(2)}\,14^{(1)}\,10^{(2)}\,6^{(3)}\,2^{(1)} \xrightarrow{\text{ex}} 26^{(1)}\,16^{(1)}\,8^{(1)}\,16^{(2)}\,8^{(2)}\,4^{(3)}\ . \tag{23}$$

The multiple partition thus obtained is

$$\lambda^{(1)} = (26,16,8), \quad \lambda^{(2)} = (16,8), \quad \lambda^{(3)} = (4). \tag{24}$$

The inverse operation, $\Lambda^{[k-1]} \to \mathrm{LP}^{[k-1]}$, amounts to viewing $(\lambda^{(1)},\cdots,\lambda^{(k-1)})$ as a sequence of clusters (cf. (14)) and reordering the clusters in decreasing value of the weight using (21) to ensure that the condition (17) is everywhere satisfied. As shown in [17], this is a finite process and it has a unique solution. We have thus a bijection

$$\mathrm{LP}^{[k-1]} \leftrightarrow \Lambda^{[k-1]}. \tag{25}$$

Let us return briefly to the parafermionic interpretation of the lattice path. A state of the form (7) corresponds to a multiple partition $\Lambda^{[k-1]}$ and thus to a lattice path $\mathrm{LP}^{[k-1]}$. A cluster of charge j is thus a combinatorial representation of a parafermionic mode $A^{(j)}$ and the parafermionic charge is twice the cluster charge. Moreover, if the path has total charge m and weight w, the conformal dimension of the corresponding parafermionic state is

$$h = w - h_{\text{frac}} = w - \frac{m^2}{k}\ . \tag{26}$$

Note that the exchange relation (21) is an abstract version of the parafermionic mode commutation relations. Recall that these commutations take the form of infinite sums (the so-called generalized commutations relations) [21]. The relation (21) amounts, roughly, to considering only the leading term of the two infinite strings of states. A similar relation is introduced in [5] and called a shuffle.

2.3 From the multiple partition to the RSOS path

Let us now describe a simple operation that adds m_k closely-packed clusters of charge k to the right of the sequence described by $\Lambda^{[k-1]}$. The number m_k is determined below. The closely-packed condition refers to the condition (12) with the equality sign: $\gamma_l^{(k)} = \gamma_{l+1}^{(k)} + 2k$. We thus add to $\Lambda^{[k-1]}$ the partition $\gamma^{(k)}$ with fixed parts given by

$$\gamma^{(k)} = ((2m_k-1)k,\cdots,3k,k)\ . \tag{27}$$

Because this is inserted to the right of $\Lambda^{[k-1]}$, we need to reshuffle all the parts of $\Lambda^{[k-1]}$ according to the modified version of (13) that includes the modes of type k:

$$\lambda_l^{(j)} \to \gamma_l^{(j)} = \lambda_l^{(j)} + 2jm_k\ . \tag{28}$$

The conditions on these reshuffled parts are thus

$$\gamma_l^{(j)} \geq \gamma_{l+1}^{(j)} + 2j\ , \tag{29}$$

and

$$\gamma_{m_j}^{(j)} \geq j + 2j(m_{j+1} + \cdots + m_k). \tag{30}$$

For $j = k$, these relations are satisfied as equalities. Denote the resulting set of k ordered partitions as $\Gamma^{[k]}$:

$$\Gamma^{[k]} = (\gamma^{(1)},\cdots,\gamma^{(k)}), \qquad \text{with} \qquad \gamma^{(j)} = (\gamma_1^{(j)},\cdots,\gamma_{m_j}^{(j)}). \tag{31}$$

However, for this definition to be complete, we still need to fix m_k. Once this value will be determined, we will have a unique procedure to go from $\Lambda^{[k-1]}$ to $\Gamma^{[k]}$ and back. That will establish the bijection

$$\Lambda^{[k-1]} \leftrightarrow \Gamma^{[k]}. \tag{32}$$

Let us now transform $\Gamma^{[k]}$ into a k-restricted RSOS path, denoted RSOS$^{[k]}$. The procedure is exactly the very one used to transform $\Lambda^{[k-1]}$ into a lattice path LP$^{[k-1]}$. This is the point where the number m_k gets fixed. It is determined by enforcing that the resulting path is connected, that is, to be such that all peaks are in contact in the sense that nowhere adjacent peaks get separated by a horizontal link. We thus fix m_k by requiring that the modified path (which now has peaks of charge k) is completely free of horizontal segments. This can always be ensured by the insertion of sufficiently many clusters of charge k since the size of each added cluster, $2k$, is larger than the increase in the weight of a cluster of charge j, which is $2j$, that results from the insertion of each charge k-cluster.

Let us now see how this can be made precise. Consider the substring containing only the clusters of charge j up to k. We then reorder the clusters in decreasing value of the weights by also ensuring that the path condition is everywhere satisfied. In addition, we require the rightmost peak of charge j to lie on a connected path. The length of a RSOS path is twice its charge content (and the charge of the peaks in a RSOS path can be read off exactly as for a lattice path [19]). Since the charge content of the path containing only peaks of charge $\geq j$ is

$$jm_j + (j+1)m_{j+1} + \cdots + km_k\,, \tag{33}$$

the connecting criterion for the rightmost peak of charge j translates into

$$\gamma_1^{(j)} + j \leq 2\,[\,jm_j + (j+1)m_{j+1} + \cdots + km_k\,]\,. \tag{34}$$

The peak of charge j as a whole is represented by a triangle with peak position at $\gamma_1^{(j)}$; this position as well as the following straight-down segment of the triangle, of length j, must both lie within the length that is determined by the charge content. This is the meaning of (34). If we reexpress this inequality in terms of the $\Lambda^{[k-1]}$ data, we have

$$\lambda_1^{(j)} + j + 2jm_k \leq 2\,[\,jm_j + \cdots + km_k\,]\,. \tag{35}$$

For any given value of $\lambda_1^{(j)}$, there is always a minimal value of m_k that ensures this bound. Indeed, the above condition implies

$$m_k \geq \left\lceil \frac{\lambda_1^{(j)} + j - 2\,[\,jm_j + \cdots + (k-1)m_{k-1}\,]}{2(k-j)} \right\rceil\,, \tag{36}$$

where $\lceil x \rceil$ is the smallest integer larger that x. This condition must hold for all values of j, from $k-1$ down to 1. We thus set

$$m_k = \max\,\left(\left\lceil \frac{\lambda_1^{(j)} + j - 2\,[\,jm_j + \cdots + (k-1)m_{k-1}\,]}{2(k-j)} \right\rceil, 1 \leq j \leq k-1 \right)\,. \tag{37}$$

This is the announced determination of m_k. This specification completes the proof of (32). For the example (23)-(24), we have

$$m_4 = \max\,\left(\left\lceil \frac{7}{6} \right\rceil, \left\lceil \frac{4}{4} \right\rceil, \left\lceil \frac{1}{2} \right\rceil \right) = \max\,(2,1,1) = 2\,. \tag{38}$$

Inverting the procedure, we can define the multiple partition $\Gamma^{[k]}$ as in (31), in terms of the conditions (29), (30) as well as

$$\gamma_1^{(j)} \leq -j + 2\,[\, jm_j + (j+1)m_{j+1} + \cdots + km_k\,]\ . \tag{39}$$

These conditions hold for all j, including $j = k$. In particular, the lower bound on $\gamma_{m_k}^{(k)}$, the upper bound on $\gamma_1^{(k)}$, together with the difference condition (29), imply that $\gamma^{(k)}$ is fixed to (27). This establishes the bijection

$$\Gamma^{[k]} \leftrightarrow \mathrm{RSOS}^{[k]}. \tag{40}$$

The chain of bijections (25), (32) and (40) implies the following correspondence:

$$\mathrm{LP}^{[k-1]} \leftrightarrow \mathrm{RSOS}^{[k]}. \tag{41}$$

This is our main result.

In the light of [17], where $\mathrm{LP}^{[k-1]}$ is related to a partition satisfying the condition (1), the correspondence (41) entails a direct bijection between such a partition and a RSOS path.

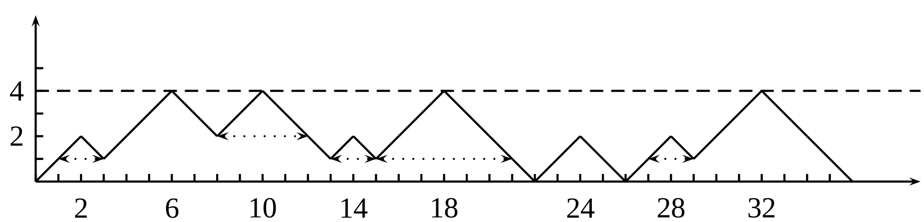
We stress that a set of k ordered partitions has an immediate interpretation in terms of parafermionic modes. For instance $\Gamma^{[k]}$ is associated to the following state

$$A^{(1)}_{-\gamma_1^{(1)}+\frac{(1+\hat{q})}{k}} \cdots A^{(1)}_{-\gamma_{m_1}^{(1)}+\frac{(1+\hat{q})}{k}} \cdots A^{(k)}_{-\gamma_1^{(k)}+\frac{(k)(k+\hat{q})}{k}} \cdots A^{(k)}_{-\gamma_{m_k}^{(k)}+\frac{(k)(k+\hat{q})}{k}}\ |0\rangle\ , \tag{42}$$

The bijection (40) thus underlies a direct combinatorial interpretation of a parafermionic quasi-particle state in terms of a RSOS path.

2.4 Examples

Figure 2: A RSOS path for $k = 4$.



Let us illustrate the map $\mathrm{RSOS}^{[k]} \to \mathrm{LP}^{[k-1]}$ by considering the $k = 4$ RSOS path of Fig. 2. Its reordering in a multiple partition denoted $\Gamma^{[4]}$ is:

$$32^{(4)}\,28^{(1)}\,24^{(2)}\,18^{(3)}\,14^{(1)}\,10^{(2)}\,6^{(4)}\,2^{(1)} \xrightarrow{\text{ex}} 30^{(1)}\,20^{(1)}\,12^{(1)}\,24^{(2)}\,16^{(2)}\,16^{(3)}\,12^{(4)}\,4^{(4)}\ . \tag{43}$$

Now delete the clusters of charge 4 and reduce the indices by inverting (28). This gives $\Lambda^{[3]}$. Then reorder the clusters to generate a lattice path:

$$26^{(1)}\,16^{(1)}\,8^{(1)}\,16^{(2)}\,8^{(2)}\,4^{(3)} \xrightarrow{\text{ex}} 26^{(1)}\,20^{(2)}\,14^{(1)}\,10^{(2)}\,6^{(3)}\,2^{(1)}\ , \tag{44}$$

which is the path of Fig. 1. This last line is the inverse of (23). Reversing this example, the first step amounts to computing the value of m_4: this is done in (38).

As a second example, let us relate the lattice paths of weight 6 for $k=3$ to their corresponding RSOS paths obtained by adding clusters of charge 3 and boosting the weights of the lower charged clusters accordingly (cf. (28)). The value of m_3 required in each case is calculated from (37) and it is indicated in every case. With m_3 fixed, the weights of $\Lambda^{[2]}$ are reshuffled appropriately. Then, if necessary, the clusters are reordered with the exchange relation (21) in order to respect the path condition (17). The rightmost sequence is the corresponding RSOS path:

$$\begin{array}{lclcl}
6^{(1)} & \xrightarrow{m_3=2} & 10^{(1)}\,9^{(3)}\,3^{(3)} & \xrightarrow{\text{ex}} & 11^{(3)}8^{(1)}\,3^{(3)} \\
6^{(2)} & \xrightarrow{m_3=2} & 14^{(2)}\,9^{(3)}\,3^{(3)} & & \\
5^{(1)}\,1^{(1)} & \xrightarrow{m_3=1} & 7^{(1)}3^{(1)}3^{(3)} & \xrightarrow{\text{ex}} & 7^{(1)}\,5^{(3)}\,1^{(1)} \\
5^{(2)}\,1^{(1)} & \xrightarrow{m_3=1} & 9^{(2)}\,3^{(1)}\,3^{(3)} & \xrightarrow{\text{ex}} & 9^{(2)}\,5^{(3)}\,1^{(1)} \\
4^{(1)}\,2^{(1)} & \xrightarrow{m_3=1} & 6^{(1)}\,4^{(1)}\,3^{(3)} & \xrightarrow{\text{ex}} & 7^{(3)}\,4^{(1)}\,2^{(1)} \\
4^{(1)}\,2^{(2)} & \xrightarrow{m_3=0} & 4^{(1)}\,2^{(2)}\;. & &
\end{array} \tag{45}$$

In the last case, the lattice path has no horizontal move, hence it remains unchanged.

3 Weighting and counting RSOS paths

3.1 The weight of a RSOS path

The weight of a RSOS path (with the understanding that our discussion pertains solely to regime II) is given in the form

$$\tilde{w} = \sum_{x=1}^{L-1} \tilde{w}(x) \qquad \text{where} \qquad w(x) = \begin{cases} \frac{x}{2} & x \text{ is an extremum of the path} \\ 0 & \text{otherwise}\;. \end{cases} \tag{46}$$

To make contact with the conformal dimension of a state, we need to subtract from this the contribution of the ground state. The latter, in turn, is specified by the total charge, say m', modulo k. If we set:

$$m' = pk + r\;, \tag{47}$$

the ground state is thus characterized by the integer r with $0 \leq r \leq k-1$. In the terminology of [14], $2r$ gives the relative charge of the parafermionic (fixed-charge modulo $2k$) submodule. In other words, the vacuum module can be broken up into r distinct submodules of different relative charge. The dimension of the highest-weight state of (parafermionic) charge $2r$ in the vacuum module is [14]:

$$h_0^{(r)} = h_{\psi^{(r)}} = \frac{r(k-r)}{k}\;. \tag{48}$$

The dimensions of the states within a given charged module all differ by integers.

Let us call gs(r) the path representing the highest-weight state of charge $2r$ in the vacuum module and by path(r) a path of total (path) charge $m' = pk+r$ for some p. The conformal dimension of the corresponding state is thus

$$h = \tilde{w}_{\text{path}(r)} - \tilde{w}_{\text{gs}(r)} + h_0^{(r)}\;. \tag{49}$$

The correspondence between a RSOS path and a parafermionic state like (42) entails a different expression for the conformal dimension. This alternative form, read off from (42), is expressed in terms of the sum of the peak x-positions, w, and the fractional dimension associated to the total charge:

$$h = w_{\mathrm{path}(r)} - h_{\mathrm{frac}} = w_{\mathrm{path}(r)} - \frac{m'^2}{k} \ . \tag{50}$$

We stress that in this expression, we sum over the position of the peaks (cf. (16)) and not half these positions and that the minima no longer contribute. Moreover, the weight of the path, corrected by h_{frac}, yields directly the conformal dimension of the corresponding state. In other words, we no longer need to subtract the ground-state contribution.

The equivalence of (49) and (50) is a direct consequence of the bijection (41). However, it is of interest to prove it directly.

We first consider the path gs(r), the ground state specified by r (so that the total charge is $m' = pk+r$). It is described by a peak of charge r at position r followed by p peaks of charge k, at positions $k+2r, 3k+2r, \dots, (2p-1)k+2r$. Its weight is simply evaluated:

$$w_{\mathrm{gs(r)}} = r + kp^2 + 2rp \ . \tag{51}$$

The first term corresponds to the weight of the first peak. The second term is $w_{\mathrm{gs}(0)}$. Finally, the third term describes the modification to the weight $w_{\mathrm{gs}(0)}$ caused by the shift in the position of the p peaks, resulting from the insertion at the beginning of the path, of a peak of height r. The expression (50) yields

$$h = w_{\mathrm{gs(r)}} - h_{\mathrm{frac}} = r + kp^2 + 2rp - \frac{(kp+r)^2}{k} = \frac{r(k-r)}{k} = h_0^{(r)}. \tag{52}$$

This agrees with the expression (49) in the case where path(r)=gs(r).

In the second step, we observe that any configuration can be obtained from gs(r) by combining the two elementary operations:

1- Keep the charge content fixed and displace a peak of charge $j \leq k-1$ toward the right by one unit.

2- Change the charge content by breaking a peak of charge n into two peaks of charge j and $n-j$.

We will verify that the modification in the weight, resulting from these two operations, is the same whether it is computed from (49) or (50), that is:

$$w - w_{\mathrm{gs(r)}} = \tilde{w} - \tilde{w}_{\mathrm{gs(r)}} \ . \tag{53}$$

Obviously, (52) and (53) imply directly the equivalence of (49) and (50):

$$h = w_{\mathrm{path}(r)} - h_{\mathrm{frac}} = (w_{\mathrm{path}(r)} - w_{\mathrm{gs}(r)}) + w_{\mathrm{gs}(r)} - h_{\mathrm{frac}} = (\tilde{w}_{\mathrm{path}(r)} - \tilde{w}_{\mathrm{gs}(r)}) + h_0^{(r)} \ .$$

It only remains to verify the equality (53) for the two basic operations. For a given charge content, the different configurations that can be generated are all obtained from a minimal-weight configuration (described in Sec. 3.3) through a sequence of basic displacements. These are described in detail in [5, 19]. (In the latter reference, the discussion pertains to the unitary minimal models; it can be applied directly to our case by reversing the direction of the displacements as discussed in Sec. 1.2). Consider then the displacement of a peak by one unit toward the right (operation (1)). The weight w is increased by 1 since the peak position is increased by 1. Similarly, $\tilde{w}$ changes by 1 since both the peak and the minimum just at its right both increase by 1. Therefore, a combination of such peak displacements always satisfy (53).

Finally, let us compare the weight of a peak with respect to that of two peaks with same total charge and in-between the same initial and final positions (operation (2)). Consider a peak of height

n centered at $x+n$. We have:

$$(n+x)^{(n)}: \qquad w=n+x, \qquad \tilde{w}=\frac{1}{2}(n+x)\;. \tag{54}$$

Compare this with the weight of the path described by a peak of height j at $j+x$ followed by a peak of weight $n-j$ at $n+j+x$:

$$(n+j+x)^{(n-j)}(j+x)^{(j)}: \qquad w=n+2j+2x \qquad \tilde{w}=\frac{1}{2}(n+4j+3x)\;, \tag{55}$$

so that

$$\Delta w=\Delta\tilde{w}=2j+x\;. \tag{56}$$

Iterating this procedure, one can break a peak into more than two peaks of lower charge and at each step we will preserve (53). This completes the proof of the equivalence of (49) and (50).

3.2 Enumerations of the RSOS paths

An intermediate step toward the derivation of a finitized form of the parafermionic character amounts to count the number of allowed configurations with fixed values of the m_j. We will relate this enumeration problem to that of counting partitions with simple restrictions. For this, let us subtract from $\gamma^{(j)}$ the following partition with m_j equal parts:

$$(\Delta_j,\cdots,\Delta_j) \qquad \text{where} \qquad \Delta_j=j+2j\,[\,m_{j+1}+\cdots+m_k\,]\;, \tag{57}$$

as well as the staircase:

$$((m_j-1)2j,\cdots,4j,2j,0)\;. \tag{58}$$

In other words, we define a new partition $\sigma^{(j)}$ by

$$\sigma_l^{(j)}=\gamma_l^{(j)}-\Delta_j-2j(m_j-l)\;. \tag{59}$$

This implies that $\sigma_{m_j}^{(j)}\geq 0$. The upper bound (39) yields

$$\begin{aligned}\sigma_1^{(j)} &\leq -j+2\,[\,jm_j+(j+1)m_{j+1}+\cdots+km_k\,]-\Delta_j-2j(m_j-1)\\ &=2\,[\,(j+1)m_{j+1}+\cdots+km_k\,]-2j\,[\,m_{j+1}+\cdots+m_k\,]\\ &=2m_{j+1}+4m_{j+2}+\cdots+2(k-j)m_k\;.\end{aligned} \tag{60}$$

We have thus found that $\sigma^{(j)}$ is a partition with at most m_j parts and whose parts are all $\leq p_j$ with p_j given by

$$p_j=2m_{j+1}+4m_{j+2}+\cdots+2(k-j)m_k. \tag{61}$$

Their enumeration is simply (omitting the upperscript (j)) [1]:

$$\sum_{\sigma_1=0}^{p_j}\sum_{\sigma_2=0}^{\sigma_1}\cdots\sum_{\sigma_{m_j}=0}^{\sigma_{m_j-1}}1=\binom{p_j+m_j}{m_j}\;. \tag{62}$$

The total number of elements $\Gamma^{[k]}$, with all m_j fixed, is thus

$$\sum_{\sigma^{(1)}}\sum_{\sigma^{(2)}}\cdots\sum_{\sigma^{(k-1)}}1=\prod_{j=1}^{k-1}\binom{p_j+m_j}{m_j}\;. \tag{63}$$

Recall that the partition $\sigma^{(k)}$ has zero parts and it is thus unique (so that it is not summed over).

For instance, consider $k=3$ and the charge content $(m_1,m_2,m_3)=(1,2,1)$, so that $\Gamma^{[3]}=\gamma_1^{(1)}\gamma_1^{(2)}\gamma_2^{(2)}\gamma_1^{(3)}$. The bounds (29), (30) and (39) give:

$$7\leq\gamma_1^{(1)}\leq 15\,,\quad \gamma_2^{(2)}\geq 6\,,\quad \gamma_1^{(2)}\leq 12\,,\quad \gamma_1^{(2)}\geq\gamma_2^{(2)}+4\,,\quad \text{and}\quad 3\leq\gamma_1^{(3)}\leq 3. \tag{64}$$

There are 9 choices for $\gamma_1^{(1)}$, 6 possible values of $(\gamma_1^{(2)},\gamma_2^{(2)})$ and $\gamma_1^{(3)}$ is fixed. That makes a total of 54 possibilities, in agreement with the combinatorial factor (63):

$$\binom{m_1+2m_2+4m_3}{m_1}\binom{m_2+2m_3}{m_2}=\binom{9}{1}\binom{4}{2}=54. \tag{65}$$

3.3 The finitized fermionic character

The generating function of all RSOS paths, which corresponds to a finitization of the $\mathcal{Z}_k$ parafermionic vacuum character, is obtained by first computing the weight of the minimal-weight configuration for a fixed charge content, evaluating the weight of the different configurations that can be obtained from this one (the number of such configurations is given by (63)) by the allowed displacements of the peaks, and finally by summing over all values of m_j compatible with a fixed value of the total charge (that is, a fixed length). The analysis is quite similar to that of [19] for the unitary minimal models and in consequence, the presentation here will be somewhat succinct.

The vacuum minimal-weight configuration (mwc(0)) for a fixed charge content (fixed values of the m_j) is the one for which all peaks are isolated (this means that their base points lie on the horizontal axis but, of course, they must be closely packed as suited for a RSOS path) and ordered by increasing value of the charge from left to right. The weight is easily found to be

$$w_{\mathrm{mwc}(0)}=m_1^2+2m_1m_2+2m_2^2+\cdots+[2m_1+\cdots+2(k-1)m+k-1]m_k+km_k^2\,, \tag{66}$$

or equivalently

$$w_{\mathrm{mwc}(0)}=\frac{1}{2}\sum_{i,j=1}^{k}r_{ij}\,m_im_j\,, \tag{67}$$

with r_{ij} defined in (18). The conformal dimension of this configuration is thus

$$h_{\mathrm{mwc}(0)}=w_{\mathrm{mwc}(0)}-\frac{m'^2}{k}=\frac{1}{2}\sum_{i,j=1}^{k}r_{ij}\,m_im_j-\frac{(m+km_k)^2}{k}=\frac{1}{2}\sum_{i,j=1}^{k-1}r_{ij}\,m_im_j-\frac{m^2}{k}. \tag{68}$$

Note the cancellation of m_k in the last step: the conformal dimension of the minimal-weight configuration is thus independent of the number of modes of type k.

Any configuration with the same charge content can be obtained from the minimal-weight configuration by a sequence of unit displacements toward the right that each increases the weight by 1. Because the length of the path is fixed, the number of such displacements is bounded: the maximal displacement of a peak of charge j is p_j. Taken into account, the weight modification transforms the binomial coefficient (69) into a q-binomial [1]:

$$\sum_{\sigma_1=0}^{p_j}\sum_{\sigma_2=0}^{\sigma_1}\cdots\sum_{\sigma_{m_j}=0}^{\sigma_{m_j-1}}q^{\sigma_1+\cdots+\sigma_{m_j}}=\begin{bmatrix}p_j+m_j\\ m_j\end{bmatrix}\,. \tag{69}$$

where

$$\begin{bmatrix}m\\ n\end{bmatrix}=\frac{(q)_m}{(q)_n(q)_{m-n}}\qquad\text{with}\qquad (q)_n=\prod_{i=1}^{n}(1-q^i)\,. \tag{70}$$

Therefore, the product of binomial coefficients (63) is transformed into the product of q-binomials:

$$\prod_{j=1}^{k-1} \binom{p_j+m_j}{m_j} \quad \rightarrow \quad \prod_{j=1}^{k-1} \begin{bmatrix} p_j+m_j \\ m_j \end{bmatrix} , \tag{71}$$

The finitized (i.e., for m' finite) vacuum character is obtained by multiplying the above number of q-weighted (relative to mwc(0)) configurations, incorporate the weight of the minimal-weight configuration and sum over all values of m_j compatible with $\sum_{j=1}^{k} jm_j = m'$:

$$\chi_0^{(m')}(q) = \sum_{\substack{m_1,\cdots,m_k\geq 0 \\ m_1+2m_2+\cdots km_k=m'}} q^{h_{\mathrm{mwc}(0)}} \prod_{j=1}^{k-1} \begin{bmatrix} p_j+m_j \\ m_j \end{bmatrix} . \tag{72}$$

The label 0 reminds that this expression refers to the vacuum module. The finitized character of the charge-r module is obtained by restricting the sum to terms with $m' \equiv r \bmod k$.

The conformal character is reproduced in the limit $m' \to \infty$ performed by taking $m_k \to \infty$. This implies that $p_j \to \infty$ for all j. Using

$$\lim_{m\to\infty} \begin{bmatrix} m \\ n \end{bmatrix} = \frac{1}{(q)_n} , \tag{73}$$

this yields

$$\chi_0(q) = \sum_{m_1,\cdots,m_{k-1}\geq 0} \frac{q^{h_{\mathrm{mwc}(0)}}}{(q)_{m_1}\cdots(q)_{m_{k-1}}} . \tag{74}$$

This is the expected result, the Lepowski-Primc (vacuum) character [18]. Again, the character of the charge-r submodule is obtained by restricting the sum to those terms which satisfy $m \equiv r \bmod k$.

4 Generalization to arbitrary irreducible modules

An irreducible module of the $\mathcal{Z}_k$ parafermionic model is characterized by its highest-weight state $|\varphi_\ell\rangle$, with $0 \leq \ell \leq k-1$ and $|\varphi_0\rangle = |0\rangle$. The charge of its highest-weight state is $\hat{q}(|\varphi_\ell\rangle) = \ell$ and its conformal dimension, h_ℓ, reads [21]:

$$h_\ell = \frac{\ell(k-\ell)}{2k(k+2)} . \tag{75}$$

Again each module can be separated into distinct submodules of fixed relative charge $2r$. The highest-weight state in the sector of relative charge $2r$ has conformal dimension [14]:

$$h_\ell^{(r)} = h_\ell + \frac{r(k-\ell-r)}{k} . \tag{76}$$

The fractional dimension of a state such as (42) of charge m' but with $|0\rangle$ replaced by $|\varphi_\ell\rangle$ is

$$h_{\mathrm{frac}} = \frac{m'(m'+\ell)}{k} . \tag{77}$$

The corresponding RSOS path starts on the vertical axis at position $(0,\ell)$ and again it is forced to finish on the x axis. Denote by gsc(ℓ,r) the ground-state configuration in the r-th sector of the module specified by ℓ. It is described as follows: the path starts with a peak of relative charge r at position r, reaches the x-axis at position $2r+\ell$ and it is completed by a sequence of $m'-r$ peaks of charge k. This construction entails the following constraint:

$$0 \leq \ell + r \leq k . \tag{78}$$

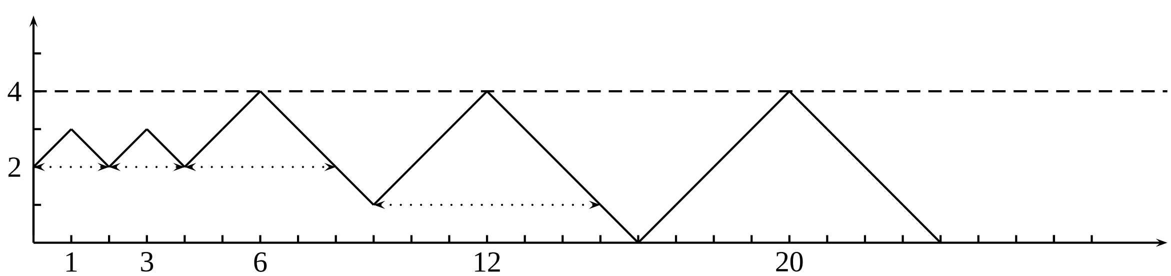
But recall that an independent set of highest-weight states of fixed charge satisfies $0 \leq r \leq k-\ell-1$ [14]. This is more restrictive than the above condition, which therefore does not entail any loss of generality. It is simple to verify that

$$h = w_{\mathrm{gsc}(\ell,r)} - h_{\mathrm{frac}} + h_\ell = h_\ell^{(r)} \,. \tag{79}$$

Consider now the modification of the minimal-weight configuration corresponding to a given set of values m_j, caused by changing the value of the initial point from $(0,0)$ to $(0,\ell)$. Let us denote this modified expression by mwc(ℓ). This configuration is described as follows. The path starts at $y=\ell$ with a sequence of m_1 peaks of charge 1 (at positions $1,3,\ldots,2m_1-1$ and note that the height of these peaks is $1+\ell$), followed by m_2 peaks of charge 2, etc. This proceeds until we reach the peaks of charge $k-\ell+1$. Since they would have height $k+1$, which is not allowed, the path, just after the sequence of $m_{k-\ell}$ peaks of charge $k-\ell$, must go down to height $\ell-1$ before describing the closely-packed sequence of peaks of charge $k-\ell+1$. The need for this extra SE link increases the weight associated to these peaks of charge $k-\ell+1$ by $m_{k-\ell+1}$ since the peak positions are displaced by one unit each as compared the minimal-weight configuration for $\ell=0$. Similarly, the peaks of charge $k-\ell+2$ must start at the height $\ell-2$ so that the straight-down segment following the last peak of charge $k-\ell+1$, which has length $k-\ell+1$ in mwc(0), has now an extra SE link. (See Fig. 3 for an example with $\ell=2$ and $k=4$.) Comparing with the minimal-weight configuration for $\ell=0$, there are two extra links before the sequence of peaks of charge $k-\ell+2$, so that all peak positions are displaced by two units; their contribution to the weight is thus augmented by $2m_{k-\ell+2}$. A similar analysis applies to all higher charge peaks. The net result is that

$$w_{\mathrm{mwc}(\ell)} = w_{\mathrm{mwc}(0)} + m_{k-\ell+1} + 2m_{k-\ell+2} + \cdots + (k-\ell)m_k. \tag{80}$$

Figure 3: The minimal-weight configuration for $k=4$ and $\ell=2$, with charge content $m_1=2$ and $m_2=m_3=m_4=1$.



The sole effect of this change on the character is to replace $h_{\mathrm{mwc}(0)}$ in (74) by $h_{\mathrm{mwc}(\ell)}$ with

$$h_{\mathrm{mwc}(\ell)} = w_{\mathrm{mwc}(\ell)} - \frac{m'(m'+\ell)}{k} \,. \tag{81}$$

The finitized character of the module with highest-weight state $|\varphi_\ell\rangle$ is therefore

$$\chi_\ell^{(m')}(q) = \sum_{\substack{m_1,\cdots,m_k\geq 0 \\ m_1+2m_2+\cdots km_k=m'}} q^{h_{\mathrm{mwc}(\ell)}} \prod_{j=1}^{k-1} \begin{bmatrix} p_j+m_j \\ m_j \end{bmatrix} \,. \tag{82}$$

Finally, we indicate the modifications on the conditions defining $\Gamma^{[k]}$ (trivially adapted from those pertaining to $\Lambda^{[k-1]}$ given in [16]): the condition (29) is unchanged but (30) and (39) are modified as

follows
$$\gamma^{(j)}_{m_j} \geq j + \max(j+\ell-k,0) + 2j(m_{j+1}+\cdots+m_k), \tag{83}$$
and
$$\gamma^{(j)}_{1} \leq -j + \max(j+\ell-k,0) + 2\left[\, jm_j + (j+1)m_{j+1}+\cdots+km_k\right], \tag{84}$$
respectively.

ACKNOWLEDGMENTS

PJ would like to thank P. Pearce and J. Rasmussen for fruitful discussions as well as the organizers of the workshop *From statistical mechanics to conformal and quantum field theory* for their hospitality. We also thank D. Ridout for a very critical reading of the manuscript and useful suggestions. The work of PJ is supported by EPSRC, while that of PM is supported by NSERC.

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